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Navigation and Meteorological Error Equations for Some Aerodynamic Parameters

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Mission Planning and Analysis Division

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SHUTTLE PROGRAM

NAVIGATION AND METEOROLOGICAL ERROR EQUATIONS FOR SOME AERODYNAMIC PARAMETERS

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CONTENTS

I. Preface

The mathematical equations in this document form the basis of a computer program used to perform an analysis of the errors that can be expected in a set of post-flight aerodynamic parameters. These errors are due to inaccuracies in the Shuttle best estimate trajectory (BET) and in the meteorological data to be provided by the National Weather Service in support of the OFT flights.

II. Introduction

The purpose of this report is to show how to start with a given state vector, $\underline{X}$, and its associated error covariance matrix, $C_{\underline{X}}$, and from these calculate the parameter vector, $\underline{Z}$, and its associated error covariance matrix, $C_{\underline{Z}}$.

We have

$$\underline{X} = \begin{bmatrix} \underline{R}_{ECI} \\ \dot{\underline{R}}_{ECI} \\ \underline{\theta}_P \\ \dot{\underline{R}}_{ATM,TOP} \\ \rho \\ T \end{bmatrix} \begin{array}{l} \text{shuttle (vehicle) ECI position vector (meters) = (m)} \\ \text{shuttle ECI velocity vector (m/sec)} \\ \text{IMU platform misalinement angles (radians) = (rad)} \\ \text{wind velocity in topodetic coordinates (m/sec)} \\ \text{atmospheric density (kg/m}^3\text{)} \\ \text{atmospheric temperature (°K),} \end{array}$$

and $C_{\underline{X}}$ is a 14×14 matrix. The source of $\underline{X}$ and $C_{\underline{X}}$ will be the following:

$$\underline{X} = \begin{bmatrix} \underline{R}_{ECI} \\ \dot{\underline{R}}_{ECI} \\ \underline{\theta}_P \\ \dot{\underline{R}}_{ATM,TOP} \\ \rho \\ T \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{postflight best estimate trajectory (BET)} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{National Meteorological Center (NMC)} \end{array}$$

the upper left hand 9×9 submatrix of $C_{\underline{X}}$ by the BET, the lower right hand 5×5 submatrix of $C_{\underline{X}}$ by NMC, and the rest are zeros; i.e.,

$$C_{\underline{X}} = \begin{pmatrix} \begin{pmatrix} BET \\ 9\times 9 \end{pmatrix} & \begin{pmatrix} 0 \\ 9\times 5 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 5\times 9 \end{pmatrix} & \begin{pmatrix} NMC \\ 5\times 5 \end{pmatrix} \end{pmatrix}$$

The parameter matrix, $\underline{Z}$, is defined as

$$\underline{Z} = \begin{bmatrix} V_T \\ \gamma_R \\ \psi_R \\ H \\ \bar{q} \\ V_{EQ} \\ M_\infty \\ \bar{V}'_\infty \\ \alpha \\ \beta \\ \beta' \\ \bar{q}\alpha \\ \bar{q}\beta \\ \bar{q}\beta' \end{bmatrix}
\begin{array}{l} \text{vehicle airspeed (m/sec)} \\ \text{flightpath angle wrt atmosphere (rad)} \\ \text{azimuth angle wrt atmosphere (rad)} \\ \text{altitude above the Fischer ellipsoid (m)} \\ \text{dynamic pressure (newtons/m}^2\text{) = (nt/m}^2\text{)} \\ \text{equivalent velocity (read by pilot) (m/sec)} \\ \text{Mach number (-)} \\ \text{hypersonic viscous parameter (-)} \\ \text{angle of attack (rad)} \\ \text{sideslip angle (stability axis) (rad)} \\ \text{slidslip angle (body axis) (rad)} \\ \text{pitch dynamic pressure (nt rad/m}^2\text{)} \\ \text{yaw dynamic pressure (stability axis) (nt rad/m}^2\text{)} \\ \text{yaw dynamic pressure (body axis) (nt rad/m}^2\text{)} \end{array}$$

All calculations in this report will be in the metric system.

III. Calculations

$C_{\underline{Z}}$ is calculated by using the identity

$$C_{\underline{Z}} = PC_{\underline{X}}P^T$$

where $P = \frac{\partial \underline{Z}}{\partial \underline{X}}$ is the 14×14 matrix of partials $\left(P_{ij} = \frac{\partial Z_i}{\partial X_j}\right)$. (2)

(C_Z is also 14×14). So our problem is reduced to calculating $\underline{Z}$ and P.

<u>Notation:</u> ECI = Earth Centered Inertial

EF = Earth Fixed (non-inertial)

TOP = Topodetic (non-inertial)

V = Vehicle

ATM = Atmosphere

B = Body

P = Platform

MP = Misalined Platform

<u>Coordinate Transformation :</u>

$$\underline{R}_{EF} = (T4)\ \underline{R}_{ECI} \tag{3}$$

where (T4) = (ROT) (RNP), (4)

$$(ROT) = \begin{bmatrix} \cos \omega_E (T-T_o) & \sin \omega_E (T-T_o) & 0 \\ -\sin \omega_E (T-T_o) & \cos \omega_E (T-T_o) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

(accounts for rotation of earth),

(RNP) is the rotation, nutation, and precession matrix, T_o is its time tag, and

$\omega_E = 7.2921159 \times 10^{-5}$ (= radians the earth turns per sec)

Since the topodetic coordinate system is centered at the vehicle rather than the center of the earth it is necessary, when transforming points, to first translate and then rotate. However, for vectors where only direction matters (e.g., velocity, acceleration, etc.) it is only necessary to rotate coordinates to get topodetic vectors to EF vectors and back. Therefore, the following transformation is used for such vectors:

$$\underline{R}_{TOP} = (T5)\ \underline{R}_{EF} \tag{6}$$

where

$$(T5) = \begin{bmatrix} -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\phi\cos\lambda & -\cos\phi\sin\lambda & -\sin\phi \end{bmatrix} \tag{7}$$

and ϕ (geodetic latitude) and λ (geodetic longitude) are calculated as follows:

set $e = 1/298.3$ (ellipticity of earth) (8)

$R_E = 6378166$ m (radius of earth) (9)

$B_o = .0067$ (initial B) (10)

and let

$$B_{i+1} = \frac{e(2-e)R_E}{\sqrt{(X_{EF}^2 + Y_{EF}^2)/(B_i+1)^2 + (1-e)^2 Z_{EF}^2}} \tag{11}$$

for $i = 0, 1, 2, 3$

define

$$B = B_4. \tag{12}$$

Then

$$\lambda = \arctan\left(\frac{Y_{EF}}{X_{EF}}\right) \tag{13}$$

$$\phi = \arctan \frac{Z_{EF}(B+1)}{\sqrt{X_{EF}^2 + Y_{EF}^2}} \tag{14}$$

Now to calculate the actual velocity of the vehicle over the ground we work in EF coordinates.

$$\underline{R}_{EF} = (T4)\ \underline{R}_{ECI} \tag{3}$$

so

$$\dot{\underline{R}}_{EF} = (\dot{T4})\ \underline{R}_{ECI} + (T4)\ \dot{\underline{R}}_{ECI} \tag{15}$$

and expressing this in TOP coordinates

$$\dot{\underline{R}}_{TOP} \equiv (T5)\ \dot{\underline{R}}_{EF} \tag{16}$$

or

$$\dot{\underline{R}}_{TOP} = (T5)\ (\dot{T4})\ \underline{R}_{ECI} + (T5)\ (T4)\ \dot{\underline{R}}_{ECI} \tag{17}$$

or defining (see equation 20 for $(\dot{T4})$)

$$(T6) = (T5)\ (\dot{T4}) \tag{18}$$

$$(T7) = (T5)\ (T4)$$

we have

$$\underset{\substack{\uparrow\\ 3\times1\\ \text{matrix}}}{\dot{\underline{R}}_{TOP}} = (T6)\ \underline{R}_{ECI} + (T7)\ \dot{\underline{R}}_{ECI}$$

$$= \Big[\underbrace{(T6)\quad (T7)\quad (0)\quad (0)}_{\substack{3\times3\\ \text{matrices}}}\quad \underset{\substack{\uparrow\\ 3\times2\\ \text{matrix}}}{(0)}\Big]\ \underset{\substack{\uparrow\\ 14\times1\\ \text{matrix}}}{\underline{X}}\ . \tag{19}$$

In the above

$$(\dot{T4}) = (\dot{ROT})\ (RNP)$$

$$= \omega_E \begin{bmatrix} -\sin \omega_E\ (T-T_c) & \cos \omega_E\ (T-T_o) & 0 \\ -\cos \omega_E\ (T-T_o) & -\sin \omega_E\ (T-T_o) & 0 \\ 0 & 0 & 0 \end{bmatrix} (RNP) \quad . \tag{20}$$

So from the above we have

$$\underbrace{\dot{\underline{R}}_{V|ATM,TOP}}_{3\times1} \equiv \text{vehicle TOP velocity wrt atmosphere}$$

$$= \dot{\underline{R}}_{TOP} - \dot{\underline{R}}_{ATM,TOP}$$

$$= \Big[\underbrace{(T6)\quad (T7)\quad (0)\quad (-I)}_{3\times3}\quad \underbrace{(0)}_{3\times2}\Big]\ \underbrace{\underline{X}}_{14\times1} \tag{21}$$

And hence

$$\frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}} = \Big[\underbrace{(T6)\quad (T7)\quad (0)\quad (-I)}_{3\times3}\quad \underbrace{(0)}_{3\times2}\Big]$$

$$+ \underbrace{\underbrace{\left[\frac{\partial}{\partial \underline{X}}\ \underbrace{\Big[(T6)\quad (T7)\quad (0)\quad (-I)\quad (0)\Big]}_{3_1\times14_2}\right]}_{3_1\times14_2\times14_3}\ \underline{X}}_{3_1\times14_3}$$

$$= \begin{bmatrix} (T6) & (T7) & (0) & (-I) & (0) \end{bmatrix}$$

$$+ \begin{bmatrix} \underbrace{\frac{\partial}{\partial \underline{X}}(T6) \quad \frac{\partial}{\partial \underline{X}}(T7) \quad \frac{\partial}{\partial \underline{X}}(0) \quad \frac{\partial}{\partial \underline{X}}(-I)}_{3\times[illegible]\times 14} & \underbrace{\frac{\partial}{\partial \underline{X}}(0)}_{3\times 2\times 14} \end{bmatrix} \underline{X}$$

$$= \begin{bmatrix} (T6) & (T7) & (0) & (-I) & (0) \end{bmatrix}$$

$$+ \begin{bmatrix} \left(\frac{\partial}{\partial \underline{X}}(T5)\right)(\dot{T4}) & \left(\frac{\partial}{\partial \underline{X}}(T5)\right)(T4) & \frac{\partial}{\partial \underline{X}}(0) & \frac{\partial}{\partial \underline{X}}(-I) & (0) \end{bmatrix} \underline{X}$$

$$= \begin{bmatrix} (T6) & (T7) & (0) & (-I) & (0) \end{bmatrix}$$

$$+ \begin{bmatrix} \underbrace{\left(\frac{\partial}{\partial \underline{X}}(T5)\right)(\dot{T4})}_{3\times 3\times 14} & \underbrace{\left(\frac{\partial}{\partial \underline{X}}(T5)\right)(T4)}_{3\times 3_1\times 14} & \underbrace{(0)}_{3\times 3_1\times 14} & \underbrace{(0)}_{3\times 3_1\times 14} & \underbrace{(0)}_{3\times 2_1\times 14} \end{bmatrix} \underbrace{\underline{X}}_{14_1\times 1}$$

$$= \begin{bmatrix} (T6) & (T7) & (0) & (-I) & (0) \end{bmatrix}$$

$$+ \begin{bmatrix} \underbrace{\left(\frac{\partial}{\partial \lambda}(T5)\right)(\dot{T4})}_{3\times 3_1} & \underbrace{\left(\frac{\partial}{\partial \lambda}(T5)\right)(T4)}_{3\times 3_1} & \underbrace{(0) \quad (0) \quad (0)}_{3\times 8_1} \end{bmatrix}^{\nabla} \underbrace{\underline{X}}_{14_1\times 1} \underbrace{\frac{\partial \lambda}{\partial \underline{X}}}_{1\times 14}$$

$$+ \begin{bmatrix} \underbrace{\left(\frac{\partial}{\partial \phi}(T5)\right)(\dot{T4})}_{3\times 3_1} & \underbrace{\left(\frac{\partial}{\partial \phi}(T5)\right)(T4)}_{3\times 3_1} & \underbrace{(0) \quad (0)}_{3\times 3_1} & (0) \end{bmatrix} \underbrace{\underline{X}}_{14_1\times 1} \underbrace{\frac{\partial \phi}{\partial \underline{X}}}_{1\times 14}$$

so

$$\frac{\partial \dot{\underline{R}}_V \big|_{ATM,\,TOP}}{\partial \underline{X}} = \begin{bmatrix} (T6) & (T7) & (0) & (-I) & (0) \end{bmatrix}$$

$$\underbrace{\underbrace{+\Big[\underbrace{\left(\frac{\partial}{\partial\lambda}(T5)\right)(\dot{T4})}_{3\times 3_1}\ \underbrace{\left(\frac{\partial}{\partial\lambda}(T5)\right)(T4)}_{3\times 3_1}\ \underbrace{(0)\quad(0)\quad(0)}_{3\times 8_1}\Big]\ \underset{14_1\times 1_2}{\underset{\uparrow}{\underline{X}}}}_{3\times 1_2}\ \underset{1_2\times 14}{\underset{\uparrow}{\frac{\partial\lambda}{\partial\underline{X}}}}}_{3\times 14}$$

$$+\Big[\underbrace{\left(\frac{\partial}{\partial\phi}(T5)\right)(\dot{T4})}_{3\times 3_1}\ \underbrace{\left(\frac{\partial}{\partial\phi}(T5)\right)(T4)}_{3\times 3_1}\ \underbrace{(0)\quad(0)\quad(0)}_{3\times 8_1}\Big]\ \underset{14_1\times 1_2}{\underset{\uparrow}{\underline{X}}}\ \underset{1_2\times 14}{\underset{\uparrow}{\frac{\partial\phi}{\partial\underline{X}}}} \tag{22}$$

where (T6) and (T7) are defined in equation (18),

(T4) in equation (4),

$(\dot{T4})$ in equation (20),

(T5) in equation (7),

λ in equation (13),

ϕ in equation (14),

$$\frac{\partial}{\partial\lambda}(T5) = \begin{bmatrix} \sin\phi\sin\lambda & -\sin\phi\cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ \cos\phi\sin\lambda & -\cos\phi\cos\lambda & 0 \end{bmatrix}, \tag{23}$$

$$\frac{\partial}{\partial\phi}(T5) = \begin{bmatrix} -\cos\phi\cos\lambda & -\cos\phi\sin\lambda & -\sin\phi \\ 0 & 0 & 0 \\ \sin\phi\cos\lambda & \sin\phi\sin\lambda & -\cos\phi \end{bmatrix}, \tag{24}$$

$$\frac{\partial \lambda}{\partial \underline{X}} = \frac{\partial \lambda}{\partial \underline{R}_{EF}} \; \frac{\partial \underline{R}_{EF}}{\partial \underline{X}}$$

$$= \left[\frac{-Y_{EF}}{X^2_{EF} + Y^2_{EF}} \, , \; \frac{X_{EF}}{X^2_{EF} + Y^2_{EF}} \, , \; 0 \right] \left[T4 \;\; (0) \;\; (0) \;\; (0) \;\; (0) \right] \qquad (25)$$

$$\frac{\partial \phi}{\partial \underline{X}} = \frac{\partial \phi}{\partial \underline{R}_{EF}} \; \frac{\partial \underline{R}_{EF}}{\partial \underline{X}}$$

NOTE: In the equation for ϕ, B is not a constant. Thus, a term including $\frac{\partial B}{\partial \underline{R}_{EF}}$ must be included.

Since B is an iterative process, it is most correct to also iterate $\frac{\partial B_n}{\partial \underline{R}_{EF}}$:

$$\frac{\partial B_n}{\partial X_{EF}} = K \left[\frac{(B_{n-1}+1)^2 2X_{EF} - (X_{EF}{}^2 + Y_{EF}{}^2) \frac{\partial B_{n-1}}{\partial X_{EF}}}{(B_{n-1}+1)^4} \right]$$

$$\frac{\partial B_n}{\partial Y_{EF}} = K \left[\frac{(B_{n-1}+1)^2 2Y_{EF} - (X_{EF}{}^2 + Y_{EF}{}^2) \frac{\partial B_{n-1}}{\partial X_{EF}}}{(B_{n-1}+1)^4} \right]$$

$$\frac{\partial B_n}{\partial Z_{EF}} = K \, 2(1-e)^2 \, Z_{EF}$$

where

$$K = \frac{-e(2-e)R_E}{2 \left[\frac{X_{EF}{}^2 + Y_{EF}{}^2}{(B_{n-1}+1)^2} + (1-e)^2 Z_{EF}{}^2 \right]^{3/2}}$$

to calculate, set $\frac{\partial B_o}{\partial \underline{R}_{EF}} = 0$ and iterate four times.

$$\frac{\partial B}{\partial \underline{R}_{EF}} = \frac{\partial B_4}{\partial \underline{R}_{EF}}$$

Thus:

$$\frac{\partial \phi}{\partial X_{EF}} = C \left[\frac{-X_{EF} Z_{EF} (B+1)}{(X_{EF}{}^2 + Y_{EF}{}^2)^{1/2}} + Z_{EF} (X_{EF}{}^2 + Y_{EF}{}^2)^{1/2} \frac{\partial B}{\partial X_{EF}} \right]$$

$$\frac{\partial \phi}{\partial Y_{EF}} = C \left[\frac{-Y_{EF} Z_{EF} (B+1)}{(X_{EF}{}^2 + Y_{EF}{}^2)^{1/2}} + Z_{EF} (X_{EF}{}^2 + Y_{EF}{}^2)^{1/2} \frac{\partial B}{\partial Y_{EF}} \right]$$

$$\frac{\partial \phi}{\partial Z_{EF}} = C \left[(X_{EF}{}^2 + Y_{EF}{}^2)^{1/2} (B+1) + (X_{EF}{}^2 + Y_{EF}{}^2)^{1/2} \frac{\partial B}{\partial Z_{EF}} \right]$$

where

$$C = \frac{1}{X_{EF}{}^2 + Y_{EF}{}^2 + Z_{EF}{}^2 (B+1)^2}$$

and

$$\frac{\partial \phi}{\partial \underline{X}} = \frac{\partial \phi}{\partial \underline{R}_{EF}} \frac{\partial \underline{R}_{EF}}{\partial \underline{X}}$$

$$= \underbrace{\left[\frac{\partial \phi}{\partial X_{EF}}, \frac{\partial \phi}{\partial Y_{EF}}, \frac{\partial \phi}{\partial Z_{EF}} \right]}_{1\times 3} \underbrace{\left[(T4) \quad (0) \quad (0) \quad (0) \quad (0) \right]}_{3\times 14} \tag{26}$$

1. V_T = vehicle airspeed = $|\dot{\underline{R}}_{V|ATM,TOP}|$ ([illegible]c) (27)

$$\frac{\partial V_T}{\partial \underline{X}} = \frac{\partial V_T}{\partial \dot{\underline{R}}_{V|ATM,TOP}} \frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}}$$

$$= \underbrace{\frac{(\dot{\underline{R}}_{V|ATM,TOP})^T}{|\dot{\underline{R}}_{V|ATM,TOP}|}}_{1\times 3} \underbrace{\frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}}}_{3\times 14} \tag{28}$$

1×14 [see equations (21), (22)]

This 1×14 matrix, $\frac{\partial V_T}{\partial \underline{X}}$ is the first row of the matrix of partials, P [see equation (1)].

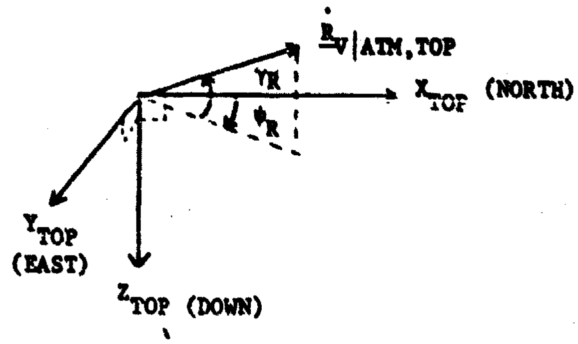

Figure 1.

2. γ_R = flightpath angle wrt atmosphere

$$= \arcsin\left[\frac{-\dot{Z}_{V|ATM,TOP}}{|\dot{\underline{R}}_{V|ATM,TOP}|}\right] \quad \text{(rad)} \tag{29}$$

$$\frac{\partial\gamma_R}{\partial\underline{X}} = \frac{\partial\gamma_R}{\partial\dot{\underline{R}}_{V|ATM,TOP}} \quad \frac{\partial\dot{\underline{R}}_{V|ATM,TOP}}{\partial\underline{X}} \tag{30}$$

$$\underset{1\times 14}{\uparrow} = \left(-1/|\dot{\underline{R}}_{V|ATM,TOP}|^2\sqrt{\dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}}\right)$$

$$\left[-\dot{X}_{V|ATM,TOP}\,\dot{Z}_{V|ATM,TOP},\ -\dot{Y}_{V|ATM,TOP}\,\dot{Z}_{V|ATM,TOP},\ \dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}\right] \underset{3X14}{\frac{\partial\dot{\underline{R}}_{V|ATM,TOP}}{\partial\underline{X}}} \tag{31}$$

3. ψ_R = azimuth angle wrt atmosphere

$$= \arctan\left[\frac{\dot{Y}_{V|ATM,TOP}}{\dot{X}_{V|ATM,TOP}}\right] \quad \text{(rad)} \tag{32}$$

$$\frac{\partial \psi_R}{\partial \underline{X}} = \frac{\partial \psi_R}{\partial \dot{\underline{R}}_{V\,ATM,TOP}} \quad \frac{\partial \dot{\underline{R}}_V|_{ATM,TOP}}{\partial \underline{X}} \tag{33}$$

$$\underset{1\times 14}{\uparrow} = \frac{1}{\dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}} \left[-\dot{Y}_{V|ATM,TOP},\ \dot{X}_{V|ATM,TOP},\ 0\right] \frac{\partial \dot{\underline{R}}_V|_{ATM,TOP}}{\underset{3X14}{\partial \underline{X}}} \tag{34}$$

4. H = altitude above the 1960 Fischer ellipsoid

$$= \left[1 - B\frac{(1-e)^2}{e(2-e)}\right] \sqrt{(X^2_{EF} + Y^2_{EF})/(B+1)^2 + Z^2_{EF}} \quad (m) \tag{35}$$

where e and B are defined above (see page 3).

For the error calculation we use the following approximation for H, which is accurate to within 1 meter for altitudes up to 200,000 meters!

$$H = |\underline{R}_{ECI}| - \frac{R_E(1-e)|\underline{R}_{ECI}|}{\sqrt{(1-e)^2(X^2_{EF} + Y^2_{EF}) + Z^2_{EF}}}$$

but we know that $|\underline{R}_{ECI}| = |\underline{R}_{EF}|$. The equation then becomes:

$$H = |\underline{R}_{EF}| - \frac{R_e(1-e)|\underline{R}_{EF}|}{\sqrt{(1-e)^2(X_{EF}{}^2 + Y_{EF}{}^2) + Z_{EF}{}^2}}$$

So

$$\frac{\partial H}{\partial X_{EF}} = \frac{X_{EF}}{|\underline{R}_{EF}|} - \frac{(1-e)[1-(1-e)^2]R_E\, Z^2_{EF}}{\left[(1-e)^2(X^2_{EF} + Y^2_{EF}) + Z^2_{EF}\right]^{3/2}} \frac{X_{EF}}{|\underline{R}_{EF}|}$$

$$\frac{\partial H}{\partial Y_{EF}} = \frac{Y_{EF}}{|\underline{R}_{EF}|} - \frac{(1-e)[1-(1-e)^2]R_E\, Z^2_{EF}}{\left[(1-e)^2(X^2_{EF} + Y^2_{EF}) + Z^2_{ECI}\right]^{3/2}} \frac{Y_{EF}}{|\underline{R}_{EF}|} \tag{37}$$

$$\frac{\partial H}{\partial Z_{EF}} = \frac{Z_{EF}}{|\underline{R}_{EF}|} + \frac{(1-e)[1-(1-e)^2]R_E(X^2_{EF} + Y^2_{EF})}{\left[(1-e)^2(X^2_{EF} + Y^2_{EF}) + Z^2_{EF}\right]^{3/2}} \frac{Z_{EF}}{|\underline{R}_{EF}|}$$

and for small e we have

$$\frac{\partial H}{\partial X_{EF}} \approx \frac{X_{EF}}{|\underline{R}_{EF}|}\left[1 - 2e\frac{R_E\, Z^2_{EF}}{|\underline{R}_{EF}|^3}\right]$$

$$\frac{\partial H}{\partial Y_{EF}} \approx \frac{Y_{EF}}{|\underline{R}_{EF}|}\left[1 - 2e\frac{R_E\, Z^2_{EF}}{|\underline{R}_{EF}|^3}\right] \tag{38}$$

$$\frac{\partial H}{\partial Z_{EF}} \approx \frac{Z_{EF}}{|\underline{R}_{EF}|}\left[1 + 2e\frac{R_E\,(X^2_{EF} + Y^2_{EF})}{|\underline{R}_{EF}|^3}\right]$$

and in fact e = 1/298.3 implies that

$$2e\frac{R_E\, Z^2_{EF}}{|\underline{R}_{EF}|^3} < \frac{1}{149}, \quad \text{and} \quad 2e\frac{R_E\,(X^2_{EF} + Y^2_{EF})}{|\underline{R}_{EF}|^3} < \frac{1}{149} \tag{39}$$

so that as claimed above the second term is small compared to the first term.

Therefore,

$$\frac{\partial H}{\partial \underline{X}} = \frac{\partial H}{\partial \underline{R}_{EF}} \frac{\partial \underline{R}_{EF}}{\partial \underline{X}}$$

$$= \underbrace{\left[\frac{\partial H}{\partial X_{EF}}, \frac{\partial H}{\partial Y_{EF}}, \frac{\partial H}{\partial Z_{EF}}\right]}_{1\times 3} \underbrace{\left[(T4) \quad (0) \quad (0) \quad (0) \quad (C)\right]}_{3\times 14}$$

where $\frac{\partial H}{\partial \underline{R}_{EF}}$ taken from equation (38) is close enough.

5. $\bar{q}$ = dynamic pressure = $\frac{1}{2}\rho V_T^2$ (nt/m^2) (41)

$$\frac{\partial \bar{q}}{\partial \underline{X}} = \bar{q}\left[\frac{1}{\rho}\frac{\partial \rho}{\partial \underline{X}} + \frac{2}{V_T}\frac{\partial V_T}{\partial \underline{X}}\right]$$

$$\underbrace{\quad}_{1\times 14} = \bar{q}\left[\frac{1}{\rho}\left(\underbrace{(0) \quad (0) \quad (0) \quad (0)}_{1\times 12} \quad \underbrace{(1,0)}_{1\times 2}\right) + \frac{2}{V_T}\underbrace{\frac{\partial V_T}{\partial \underline{X}}}_{1\times 14}\right] \tag{42}$$

(see equation (28) for $\frac{\partial V_T}{\partial \underline{X}}$

Here we neglect terms like

$\frac{\partial \rho}{\partial \underline{R}_{ECI}}$ since they are small compared to $\frac{\partial V_T}{\partial \underline{X}}$, etc.

6. V_{EQ} = equivalent velocity $= \sqrt{\frac{2\bar{q}}{\rho_o}}$ (m/sec) (43)

where ρ_o = atmospheric density at sea level

$= 1.2250\ kg/m^3$ (1962, standard atmosphere) (44)

$$\frac{\partial V_{EQ}}{\partial \underline{X}} = \frac{V_{EQ}}{2\bar{q}} \frac{\partial \bar{q}}{\partial \underline{X}} \tag{45}$$

[see equation (42)]

7. M_∞ = Mach number = $\dfrac{V_T}{C_S} = \dfrac{V_T}{\sqrt{\gamma R T}}$ (no dimn) (46)

where C_S = speed of sound

$$\gamma = 1.40$$

$$R = 287.051803 \ \frac{m^2}{{}^{o}K\text{-}sec^2} \tag{47}$$

so that

$$\gamma R = 401.872524 \ \frac{m^2}{{}^{o}K\text{-}sec^2}$$

$$\underbrace{\frac{\partial M_\infty}{\partial \underline{X}}}_{1\times14} = M_\infty \left[\frac{1}{V_T} \frac{\partial V_T}{\partial \underline{X}} - \frac{1}{2T} \frac{\partial T}{\partial \underline{X}} \right]$$

$$= M_\infty \left[\frac{1}{V_T} \underbrace{\frac{\partial V_T}{\partial \underline{X}}}_{1\times14} - \frac{1}{2T} \ \underbrace{((0) \quad (0) \quad (0) \quad (0)}_{1\times12} \quad \underbrace{(0,1))}_{1\times2} \right] \tag{48}$$

8. $\overline{V}'_\infty$ = hypersonic viscous parameter

$$= M_\infty \sqrt{\frac{C'_\infty}{R_{e_\infty L_B}}} \tag{49}$$

where

$$\left.\begin{aligned} R_{e_\infty L_B} &= \frac{V_T \rho L_B}{\mu} \\ L_B &= 32.765 \ m \\ \mu &= \frac{1.458001 \times 10^{-6} \ T^{1.5}}{T + 110.4} \quad \frac{nt/sec}{m^2} \end{aligned}\right\} \tag{50}$$

and

$$C'_{\infty} = \left[\frac{T'}{T}\right]^{1.5} \left[\frac{T + 122.1 \times 10^{-(5/T)}}{T' + 122.1 \times 10^{-(5/T')}}\right] \tag{51}$$

where

$$T' = 726.97 + .468T + 3.63921 \times 10^{-5}\, V_T^{\,2} \qquad {}^{\circ}K$$

$$\frac{\partial V'_{\infty}}{\partial \underline{X}} = \bar{V}'_{\infty}\left[\frac{1}{M_{\infty}}\frac{\partial M_{\infty}}{\partial \underline{X}} + \frac{1}{2C'_{\infty}}\frac{\partial C'_{\infty}}{\partial \underline{X}} - \frac{1}{2R_{e_{\infty}L_B}}\frac{\partial R_{e_{\infty}L_B}}{\partial \underline{X}}\right]$$

$$= \bar{V}'_{\infty}\left[\frac{1}{M_{\infty}}\frac{\partial M_{\infty}}{\partial \underline{X}} + \frac{1}{2C'_{\infty}}\left[\frac{\partial C'_{\infty}}{\partial T}\frac{\partial T}{\partial \underline{X}} + \frac{\partial C'_{\infty}}{\partial T'}\frac{\partial T'}{\partial \underline{X}}\right]\right.$$

$$\left. - \frac{1}{2}\left[\frac{1}{V_T}\frac{\partial V_T}{\partial \underline{X}} + \frac{1}{\rho}\frac{\partial \rho}{\partial \underline{X}} - \frac{1}{\mu}\frac{\partial \mu}{\partial \underline{X}}\right]\right] \tag{52}$$

$$\frac{\partial C'_{\infty}}{\partial T} = C'_{\infty}\left[-\frac{1.5}{T} + \frac{1 + (\log 10)\cdot 122.1 \times 10^{-(5/T)}\left[\frac{5}{T^2}\right]}{T + 122.1 \times 10^{-(5/T)}}\right] \tag{53}$$

$$\frac{\partial C'_{\infty}}{\partial T'} = -C'_{\infty}\left[-\frac{1.5}{T'} + \frac{1 + (\log 10)\cdot 122.1 \times 10^{-(5/T')}\left[\frac{5}{T'^2}\right]}{T' + 122.1 \times 10^{-(5/T')}}\right] \tag{54}$$

$$\frac{\partial T'}{\partial \underline{X}} = .468\,\frac{\partial T}{\partial \underline{X}} + 2\cdot(3.63921 \times 10^{-5})\, V_T\,\frac{\partial V_T}{\partial \underline{X}} \tag{55}$$

$$\frac{\partial \mu}{\partial \underline{X}} = \frac{\partial \mu}{\partial T}\frac{\partial T}{\partial \underline{X}} \tag{56}$$

$$\frac{\partial \mu}{\partial T} = \mu \left[\frac{1.5}{T} - \frac{1}{T + 110.4} \right] \tag{57}$$

$$\underset{\substack{\uparrow \\ 1\times 14}}{\frac{\partial T}{\partial \underline{X}}} = \Big(\underbrace{(0) \quad (0) \quad (0) \quad (0)}_{1\times 12} \quad \underset{\substack{\uparrow \\ 1\times 2}}{(0,1)} \Big) \tag{58}$$

$$\underset{\substack{\uparrow \\ 1\times 14}}{\frac{\partial \rho}{\partial \underline{X}}} = \Big(\underbrace{(0) \quad (0) \quad (0) \quad (0)}_{1\times 12} \quad \underset{\substack{\uparrow \\ 1\times 2}}{(1,0)} \Big) \tag{59}$$

So

$$\begin{aligned}
\frac{\partial V'_\infty}{\partial \underline{X}} = \overline{V}'_\infty \Bigg\{ & \frac{1}{M_\infty} \frac{\partial M_\infty}{\partial \underline{X}} - \frac{1}{2V_T} \frac{\partial V_T}{\partial \underline{X}} - \frac{1}{2\rho} \frac{\partial \rho}{\partial \underline{X}} \\
& + \frac{1}{2} \left[\frac{1.5}{T} - \frac{1}{T + 110.4} \right] \frac{\partial T}{\partial \underline{X}} \\
& + \frac{1}{2} \left[\frac{-1.5}{T} + \frac{1 + (\log 10) \cdot 122.1 \times 10^{-(5/T)} \left[\frac{5}{T^2} \right]}{T + 122.1 \times 10^{-(5/T)}} \right] \frac{\partial T}{\partial \underline{X}} \\
& - \frac{1}{2} \left[\frac{-1.5}{T'} + \frac{1 + (\log 10) \cdot 122.1 \times 10^{-(5/T')} \left[\frac{5}{T'^2} \right]}{T' + 122.1 \times 10^{-(5/T')}} \right] \\
& \left[.468 \frac{\partial T}{\partial \underline{X}} + 2 \, (3.63921 \times 10^{-5}) \, V_T \frac{\partial V_T}{\partial \underline{X}} \right] \Bigg\}
\end{aligned} \tag{60}$$

More Coordinate Transformations:

Recall equation (15):

$$\dot{\underline{R}}_{EF} = (\dot{T4})\, \underline{R}_{ECI} + (T4)\, \dot{\underline{R}}_{ECI} \tag{15}$$

where $(\dot{T4})$ and (T4) are defined in equations (20) and (4), respectively.

Also we have

$$\dot{\underline{R}}_{ATM,EF} = (T5)^T\, \dot{\underline{R}}_{ATM,TOP} \tag{61}$$

where (T5) is defined in equation (7).

So that

$$\dot{\underline{R}}_{V|ATM,EF} = (\dot{T4})\, \underline{R}_{ECI} + (T4)\, \dot{\underline{R}}_{ECI} - (T5)^T\, \dot{\underline{R}}_{ATM,TOP} \tag{62}$$

From here we can go to body coordinates:

$$\dot{\underline{R}}_{V|ATM,B} = (Q)\ (T)\ (T_p)\ (T4)^T\, \dot{\underline{R}}_{V|ATM,EF} \tag{63}$$

where $(T4)^T$ rotates back to ECI; (T_p) is a constant matrix that transforms from ECI to nominal platform coordinates; i.e.,

$$\underline{R}_P = (T_p)\, \underline{R}_{ECI}; \tag{64}$$

(T) transforms nominal platform coordinates to actual, misalined coordinates; i.e.,

$$\underline{R}_{MP} = (T)\, \underline{R}_p, \tag{65}$$

where T is given by

$$(T) = \begin{bmatrix} C_{p2}C_{p3} & C_{p1}S_{p3} + S_{p1}S_{p2}C_{p3} & S_{p1}S_{p3} - C_{p1}S_{p2}C_{p3} \\ -C_{p2}S_{p3} & C_{p1}C_{p3} - S_{p1}S_{p2}S_{p3} & S_{p1}C_{p3} + C_{p1}S_{p2}S_{p3} \\ S_{p2} & -S_{p1}C_{p2} & C_{p1}C_{p2} \end{bmatrix} \tag{66}$$

(here $C_{p1} = \cos\theta_{p1}$, $S_{p2} = \sin\theta_{p2}$, and similarly for the others) and since the angles θ_{p1}, θ_{p2}, and θ_{p3} are small, we have for error analysis purposes: $(T) \approx (T')$ where

$$(T') = \begin{bmatrix} 1 & \theta_{p3} & -\theta_{p2} \\ -\theta_{p3} & 1 & \theta_{p1} \\ \theta_{p2} & -\theta_{p1} & 1 \end{bmatrix} \tag{67}$$

($\underline{\theta}_p$ is part of the given state vector $\underline{X}$); and where (Q) transforms the misalined platform coordinates to body coordinates; i.e.,

$$\dot{\underline{R}}_{V|ATM,B} = (Q)\ \dot{\underline{R}}_{V|ATM,MP}, \tag{68}$$

where Q is a transformation matrix formed by rotation of the platform about its axes by their respective gimbal angles.

$$\begin{aligned} \dot{\underline{R}}_{V|ATM,B} &= (Q)\ (T)\ (T_p)\ (T4)^T\ \dot{\underline{R}}_{V|ATM,EF} \\ &= (Q)\ (T)\ (T_p)\ (T4)^T \left[(\dot{T4})\ \underline{R}_{ECI} + (T4)\ \dot{\underline{R}}_{ECI} - (T5)^T\ \dot{\underline{R}}_{ATM,TOP}\right] \\ &= (Q)\ (T)\ (T_p) \left[(T4)^T(\dot{T4})\ \underline{R}_{ECI} + \dot{\underline{R}}_{ECI} - (T4)^T(T5)^T\ \dot{\underline{R}}_{ATM,TOP}\right] \end{aligned} \tag{69}$$

or defining

$$\left.\begin{aligned} (T8) &= (T4)^T(\dot{T4}) \\ (T9) &= (T4)^T(T5)^T \end{aligned}\right\} \tag{70}$$

we have

$$\dot{\underline{R}}_{V|ATM,B} = (Q)\ (T)\ (T_p)\ \left[(T8)\ \underline{R}_{ECI} + \dot{\underline{R}}_{ECI} - (T9)\ \dot{\underline{R}}_{ATM,TOP}\right] \tag{71}$$

and using here T' instead of T for the error analysis:

$$\frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}} = (Q)\ (T')\ (T_p)\ \left[(T_8)\quad (I)\quad (0)\quad (-T9)\quad (0)\right]$$

$$\left[\left(-\frac{\partial \dot{T9}}{\partial \underline{R}_{ECI}}\right)(0)\ \left(\frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{\theta}_p}\right)\quad (0)\quad (0)\right] \tag{72}$$

and $\dfrac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{\theta}_p}$ is given by:

$$\frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \theta_{p_i}} = (Q)\left(\frac{\partial T'}{\partial \theta_{p_i}}\right)(T_p)\ \left[(T_8)\ \underline{R}_{ECI} + \dot{\underline{R}}_{ECI} - (T9)\ \dot{\underline{R}}_{ATM,TOP}\right] \tag{73}$$

where

$$\left.\begin{aligned}
\frac{\partial T'}{\partial \theta_{p_1}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
\frac{\partial T'}{\partial \theta_{p_2}} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
\frac{\partial T'}{\partial \theta_{p3}} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}\right\} \tag{74}$$

where

$$\frac{\partial T9}{\partial \underline{R}_{ECI}}$$ is given by

$$\frac{\partial T9}{\partial \underline{R}_{ECI}} = (T4)^T \frac{\partial (T5)^T}{\partial \underline{R}_{ECI}} = (T4)^T \left[\frac{\partial (T5)}{\partial \underline{R}_{ECI}}\right]^T$$

and

$$\frac{\partial T9}{\partial \underline{R}_{ECI}} = (T4)^T \left[\frac{\partial (T5)}{\partial \lambda}\right]^T \frac{\partial \lambda}{\partial \underline{R}_{ECI}} + (T4)^T \left[\frac{\partial (T5)}{\partial \phi}\right]^T \frac{\partial \phi}{\partial \underline{R}_{ECI}}$$

where $\frac{\partial T5}{\partial \lambda}$ and $\frac{\partial T5}{\partial \phi}$ are given in equations (23) and (24)

and $\frac{\partial \lambda}{\partial \underline{R}_{ECI}}$ and $\frac{\partial \phi}{\partial \underline{R}_{ECI}}$ are given in equations (25) and (26).

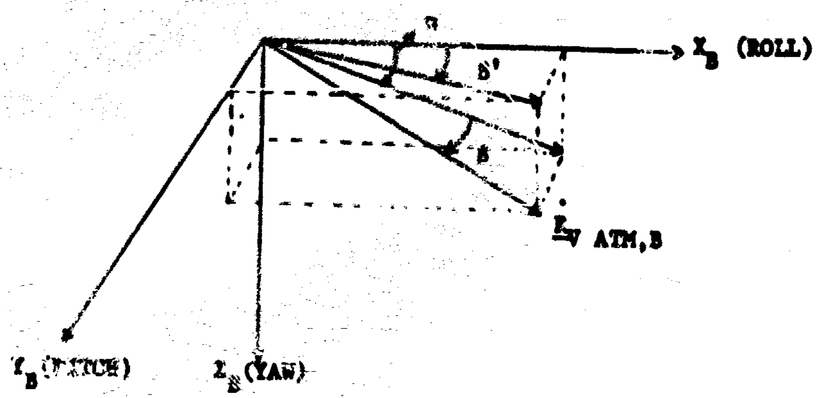

Figure 2.

9. α = angle of attack = $\arctan\left(\dfrac{\dot{z}_{V|ATM,B}}{\dot{x}_{V|ATM,B}}\right)$ (75)

$$\frac{\partial\alpha}{\partial\dot{\underline{R}}_{V|ATM,B}} = \frac{\dot{x}^2_{V|ATM,B}}{\dot{x}^2_{V|ATM,B}+\dot{z}^2_{V|ATM,B}}\ \frac{\dot{z}_{V|ATM,B}}{\dot{x}_{V|ATM,B}}\left(\frac{-1}{\dot{x}_{V|ATM,B}},\ 0\ ,\frac{1}{\dot{z}_{V|ATM,B}}\right)$$

$$= \sin\alpha\cos\alpha\left(\frac{-1}{\dot{x}_{V|ATM,B}},\ 0\ ,\ \frac{1}{\dot{z}_{V|ATM,B}}\right) \tag{76}$$

$$\frac{\partial\alpha}{\partial\underline{X}} = \frac{\partial\alpha}{\partial\dot{\underline{R}}_{V|ATM,B}}\ \frac{\partial\dot{\underline{R}}_{V|ATM,B}}{\partial\underline{X}}$$

$$= \frac{1}{\dot{x}^2_{V|ATM,B}+\dot{z}^2_{V|ATM,B}} \tag{77}$$

$$\left(-\dot{z}_{V|ATM,B},\ 0,\ \dot{x}_{V|ATM,B}\right)\frac{\partial\dot{\underline{R}}_{V|ATM,B}}{\partial\underline{X}}$$

10. β = sideslip angle (stability axis) $= \arctan\left(\dfrac{\dot{Y}_{V|ATM,B}}{\sqrt{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}}\right)$

$$(78)$$

$$\frac{\partial \beta}{\partial \dot{\underline{R}}_{V|ATM,B}} = \frac{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}{|\dot{\underline{R}}_{V|ATM,B}|^2} \frac{\dot{Y}_{V|ATM,B}}{\sqrt{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}}$$

$$\left(\frac{-\dot{X}_{V|ATM,B}}{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}},\ \frac{1}{\dot{Y}_{V|ATM,B}},\ \frac{-\dot{Z}_{V|ATM,B}}{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}\right)$$

$$= \sin\beta\cos\beta\left(\frac{-\dot{X}_{V|ATM,B}}{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}},\ \frac{1}{\dot{Y}_{V|ATM,B}},\ \frac{-\dot{Z}_{V|ATM,B}}{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}\right)$$

$$(79)$$

$$\frac{\partial \beta}{\partial \underline{X}} = \frac{\partial \beta}{\partial \dot{\underline{R}}_{V|ATM,B}} \frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}}$$

$$= \frac{1}{\sqrt{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}\left(\dot{X}^2_{V|ATM,B} + \dot{Y}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}\right)}$$

$$\left(-\dot{X}_{V|ATM,B}\,\dot{Y}_{V|ATM,B},\ \dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B},\ -\dot{Y}_{V|ATM,B}\,\dot{Z}_{V|ATM,B}\right)$$

$$\frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}} \qquad (80)$$

11. β' = sideslip angle (body axis)

$$= \arctan\left(\frac{\dot{Y}_{V|ATM,B}}{\dot{X}_{V|ATM,B}}\right) \qquad (81)$$

$$\frac{\partial\beta'}{\partial \underline{X}} = \frac{1}{\dot{X}^2_{V|ATM,B} + \dot{X}^2_{V|ATM,B}} \left(-\dot{Y}_{V|ATM,B},\ \dot{X}_{V|ATM,B},\ 0\right) \frac{\partial \underline{\dot{R}}_{V|ATM,B}}{\partial \underline{X}} \qquad (82)$$

12. $\bar{q}\alpha$ = pitch dynamic pressure (83)

$$\frac{\partial(\bar{q}\alpha)}{\partial \underline{X}} = \bar{q}\frac{\partial\alpha}{\partial \underline{X}} + \alpha\frac{\partial\bar{q}}{\partial \underline{X}} \qquad (84)$$

13. $\bar{q}\beta$ = yaw dynamic pressure (stability axis) (85)

$$\frac{\partial(\bar{q}\beta)}{\partial \underline{X}} = \bar{q}\frac{\partial\beta}{\partial \underline{X}} + \beta\frac{\partial\bar{q}}{\partial \underline{X}} \qquad (86)$$

14. $\bar{q}\beta'$ = yaw dynamic pressure (body axis) (87)

$$\frac{\partial(\bar{q}\beta')}{\partial \underline{X}} = \bar{q}\frac{\partial\beta'}{\partial \underline{X}} + \beta'\frac{\partial\bar{q}}{\partial \underline{X}} \qquad (88)$$

IV. Summary of Equations

Given $\underline{X} = \begin{bmatrix} \underline{R}_{ECI} \\ \underline{\dot{R}}_{ECI} \\ \underline{\theta}_p \\ \underline{\dot{R}}_{ATM,TOP} \\ \rho \\ T \end{bmatrix}$ and $C_{\underline{X}}$.

$(RNP) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ 3×3 matrix, T_o

$$(ROT) = \begin{bmatrix} \cos \omega_E (T-T_o) & \sin \omega_E (T-T_o) & 0 \\ -\sin \omega_E (T-T_o) & \cos \omega_E (T-T_o) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

$$(T4) = (ROT)\ (RNP) \quad (4)$$

$$\underline{R}_{EF} = (T4)\ \underline{R}_{ECI} \quad (3)$$

$$e = 1/298.3 \quad (8)$$

$$R_E = 6378166 \text{ meters} \quad (9)$$

$$B_o = .0067 \quad (10)$$

$$B_{i+1} = \frac{e(2-e)\ R_E}{\sqrt{(X_{EF}^2 + Y_{EF}^2)/(B_i+1)^2 + (1-e)^2 Z_{EF}^2}}, \quad i \geq 0 \quad (11)$$

$$B = B_4 \quad (12)$$

$$\lambda = \arctan \frac{Y_{EF}}{X_{EF}} \quad (13)$$

$$\phi = \arctan\left(\frac{Z_{EF}\ (B+1)}{\sqrt{X_{EF}^2 + Y_{EF}^2}}\right) \quad (14)$$

$$(T5) = \begin{bmatrix} -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ -\sin \lambda & \cos \lambda & 0 \\ -\cos \phi \cos \lambda & -\cos \phi \sin \lambda & -\sin \phi \end{bmatrix} \quad (7)$$

$$(\dot{T4}) = \omega_E \begin{bmatrix} -\sin \omega_E (T-T_o) & \cos \omega_E (T-T_o) & 0 \\ -\cos \omega_E (T-T_o) & -\sin \omega_E (T-T_o) & 0 \\ 0 & 0 & 0 \end{bmatrix} (RNP) \quad (20)$$

$$\left.\begin{aligned}(T6) &= (T5)\ (\ddot{T4}) \\ (T7) &= (T5)\ (T4)\end{aligned}\right\} \tag{18}$$

$$\underset{\substack{\uparrow \\ 3\times 14}}{\frac{\partial \dot{R}_{V|ATM,TOP}}{\partial \underline{X}}} = \Big[\underbrace{(T6) \quad (T7) \quad (0) \quad (-I)}_{3\times 3} \quad \underset{\substack{\uparrow \\ 3\times 2}}{(0)}\Big]$$

$$+\left[\underbrace{\left[\left[\left[\frac{\partial}{\partial\lambda}(T5)\right](\dot{T4})\right]\left[\left[\frac{\partial}{\partial\lambda}(T5)\right](T4)\right] \quad (0) \quad (0) \quad (0)\right]}_{3\times 14} \underset{\substack{\uparrow \\ 14\times 1}}{\underline{X}}\right] \underset{\substack{\uparrow \\ 1\times 14}}{\frac{\partial\lambda}{\partial \underline{X}}}$$

$$+\left[\underbrace{\left[\left[\left[\frac{\partial}{\partial\phi}(T5)\right](\dot{T4})\right]\left[\left[\frac{\partial}{\partial\phi}(T5)\right](T4)\right] \quad (0) \quad (0) \quad (0)\right]}_{3\times 14} \underset{\substack{\uparrow \\ 14\times 1}}{\underline{X}}\right] \underset{\substack{\uparrow \\ 1\times 14}}{\frac{\partial\phi}{\partial \underline{X}}} \tag{22}$$

$$\dot{\underline{R}}_{V|ATM,TOP} = \Big[(T6) \quad (T7) \quad (0) \quad (-I) \quad (0)\Big]\ \underline{X} \tag{21}$$

$$\rho_o = 1.2250\ \text{kg/m}^3 \tag{44}$$

$$\gamma R = 401.872524\ \text{m}^2/^\circ\text{K-sec}^2 \tag{47}$$

$$\left.\begin{aligned} L_B &= 32.765\ \text{m} \\ \mu &= \frac{1.458001 \quad 10^{-6} T^{1.5}}{T + 110.4}\ \frac{\text{nt/sec}}{\text{m}^2}\end{aligned}\right\} \tag{50}$$

$$(T_p) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$ **REFSMMAT** 3×3 matrix such that

$$\underline{R}_p = (T_p)\, \underline{R}_{ECI} \tag{64}$$

T_p is given after the flight.

$$(T) = \begin{bmatrix} C_{p2}C_{p3} & C_{p1}S_{p3} + S_{p1}S_{p2}C_{p3} & S_{p1}S_{p3} - C_{p1}S_{p2}C_{p3} \\ -C_{p2}S_{p3} & C_{p1}C_{p3} - S_{p1}S_{p2}S_{p3} & S_{p1}C_{p3} + C_{p1}S_{p2}S_{p3} \\ S_{p2} & -S_{p1}C_{p2} & C_{p1}C_{p2} \end{bmatrix} \tag{66}$$

$(C_{p1} \equiv \cos\theta_{p1},\ S_{p1} \equiv \sin\theta_{p1},\ \text{etc.})$

$$Q = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$ 3×3 matrix [see discussion, page 18 after equation (68)]

$$\left.\begin{aligned} (T8) &= (T4)^T(\dot{T4}) \\ (T9) &= (T4)^T(T5)^T \end{aligned}\right\} \tag{70}$$

$$\underline{V} = (T_p)\left[(T8)\,\underline{R}_{ECI} + \dot{\underline{R}}_{ECI} - (T9)\,\dot{\underline{R}}_{ATM,TOP}\right] \tag{89}$$

$$\dot{\underline{R}}_{V|ATM,B} = (Q)\,(T)\,\underline{V} \tag{71}$$

$$\left.\begin{aligned} \frac{\partial T'}{\partial\theta_{p1}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ \frac{\partial T'}{\partial\theta_{p1}} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{74}$$

$$\frac{\partial T'}{\partial \theta_{p3}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\underline{R}}_V|_{ATM,B}}{\partial \theta_{p_i}} = (Q) \left[\frac{\partial T}{\partial \theta_{p_i}}\right] \underline{V} \tag{73}$$

$$\frac{\partial \dot{\underline{R}}_V|_{ATM,B}}{\partial \underline{\theta}_p} = \underbrace{\left(\frac{\dot{\underline{R}}_V|_{ATM,B}}{\partial \theta_{p1}}, \frac{\dot{\underline{R}}_V|_{ATM,B}}{\partial \theta_{p2}}, \frac{\dot{\underline{R}}_V|_{ATM,B}}{\partial \theta_{p3}}\right)}_{\substack{3\times 1 \\ \text{matrices}}} \tag{70}$$

$$(T') = \begin{bmatrix} 1 & \theta_{p3} & -\theta_{p2} \\ -\theta_{p3} & 1 & \theta_{p1} \\ \theta_{p2} & -\theta_{p1} & 1 \end{bmatrix} \tag{67}$$

$$\frac{\partial T9}{\partial \underline{R}_{ECI}} = (T4)^T \frac{\partial}{\partial \lambda} (T5)^T \frac{\partial \lambda}{\partial R_{ECI}} + (T4)^T \frac{\partial}{\partial \phi} (T5)^T \frac{\partial \phi}{\partial \underline{R}_{ECI}}$$

$$\underset{3\times 14}{\frac{\partial \underline{R}_V|_{ATM,B}}{\partial \underline{X}}} = \underset{3\times 3}{\left[(Q)\ (T')\ (T_p)\right]} \underset{3\times 2}{\left[(T8)\ (I)\ (0)\ (-T9)\ (0)\right]} \tag{72}$$

$$+ \left[\frac{\partial T9}{\partial \underline{R}_{ECI}} \quad (0) \quad \frac{\partial \underline{R}_V|_{ATM,B}}{\partial \Phi} \quad (0) \quad (0)\right]$$

$$Z_1 = V_T = \left|\dot{R}_{V|ATM,TOP}\right| \tag{27}$$

$$Z_2 = \gamma R = \arcsin\left(\frac{-\dot{Z}_{V|ATM,TOP}}{Z_1}\right) \tag{29}$$

$$Z_3 = \psi R = \arctan\left(\frac{\dot{Y}_{V|ATM,TOP}}{\dot{X}_{V|ATM,TOP}}\right) \tag{32}$$

$$Z_4 = H = \left[1 - B\,\frac{(1-e)^2}{e(2-e)}\right] \sqrt{(X_{EF}^2 + Y_{EF}^2)/(B+1)^2 + Z_{EF}^2} \tag{35}$$

$$Z_5 = \bar{q} = \frac{1}{2}\,\rho\, Z_1^2 \tag{41}$$

$$Z_6 = \sqrt{\frac{2Z_5}{\rho_o}} = V_{EQ} \tag{43}$$

$$Z_7 = M_\infty = \frac{Z_1}{\sqrt{(\gamma R)T}} \tag{46}$$

$$\left.\begin{aligned} T' &= 226.97 + .468T + (3.63921 \times 10^{-5})\, Z_1^2 \\ C' &= \left[\frac{T'}{T}\right]^{1.5} \left[\frac{T + 122.1 \times 10^{-(5/T)}}{T' + 122.1 \times 10^{-(5/T')}}\right] \end{aligned}\right\} \tag{51}$$

$$R_{e_{\infty L_B}} = \frac{Z_1 \rho L_B}{\mu} \tag{50}$$

$$Z_8 = \bar{V}'_\infty = Z_7 \sqrt{\frac{C'_\infty}{R_{e_{\infty L_B}}}} \tag{49}$$

$$Z_9 = \alpha = \arctan\left(\frac{\dot{Z}_{V|ATM,B}}{\dot{X}_{V|ATM,B}}\right) \tag{75}$$

$$Z_{10} = \beta = \arctan\left(\frac{\dot{Y}_{V|ATM,B}}{\sqrt{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}}}\right) \tag{78}$$

$$Z_{11} = \beta' = \arctan\left(\frac{\dot{Y}_{V|ATM,B}}{\dot{X}_{V|ATM,B}}\right) \tag{81}$$

$$Z_{12} = \bar{q}\alpha = Z_5 Z_9 \tag{83}$$

$$Z_{13} = \bar{q}\beta = Z_5 Z_{10} \tag{85}$$

$$Z_{14} = \bar{q}\beta' = Z_5 Z_{11} \tag{87}$$

$$\frac{\partial Z_1}{\partial \underline{X}} = \frac{\partial V_T}{\partial \underline{X}} = \frac{\dot{\underline{R}}_{V|ATM,TOP})^T}{Z_1} \frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}} \tag{28}$$

$$\frac{\partial Z_2}{\partial \underline{X}} = \frac{\partial \gamma_R}{\partial \underline{X}} = \left(-1/|\dot{R}_{V|ATM,TOP}|^2 \sqrt{\dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}}\right) \left[-\dot{X}_{V|ATM,TOP}\ \dot{Z}_{V|ATM,TOP},\ -\dot{Y}_{V|ATM,TOP}\ \dot{Z}_{V|ATM,TOP},\ \dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}\right] \underbrace{\frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}}}_{3X14} \tag{31}$$

$$\frac{\partial Z_3}{\partial \underline{X}} = \frac{\partial \psi_R}{\partial \underline{X}} = \frac{1}{\dot{X}^2_{V|ATM,TOP} + \dot{Y}^2_{V|ATM,TOP}} \left[-\dot{Y}_{V|ATM,TOP},\ \dot{X}_{V|ATM,TOP},\ 0\right] \underbrace{\frac{\partial \dot{\underline{R}}_{V|ATM,TOP}}{\partial \underline{X}}}_{3X14} \tag{34}$$

$$\left.\begin{aligned}
\frac{\partial H}{\partial X_{EF}} &= \frac{X_{EF}}{|\underline{R}_{EF}|}\left[1 - 2e\,\frac{R_E Z^2_{EF}}{|\underline{R}_{EF}|^3}\right] \\
\frac{\partial H}{\partial Y_{EF}} &= \frac{Y_{EF}}{|\underline{R}_{EF}|}\left[1 - 2e\,\frac{R_e Z^2_{EF}}{|\underline{R}_{EF}|^3}\right] \\
\frac{\partial H}{Z_{EF}} &= \frac{Z_{EF}}{\underline{R}_{EF}}\left[1 + 2e\,\frac{R_e(X^2_{EF} + Y^2_{EF})}{|\underline{R}_{EF}|^3}\right]
\end{aligned}\right\} \tag{38}$$

$$\frac{\partial Z_4}{\partial \underline{X}} = \frac{\partial H}{\partial \underline{X}} = \underset{1\times 3}{\left[\frac{\partial H}{\partial X_{EF}},\ \frac{\partial H}{\partial Y_{EF}},\ \frac{\partial H}{\partial Z_{EF}}\right]}\ \underset{3\times 14}{\left[(T4)\quad (0)\quad (0)\quad (0)\quad (0)\right]} \tag{40}$$

$$\frac{\partial Z_5}{\partial \underline{X}} = \frac{\partial \bar{q}}{\partial \underline{X}} = Z_5\left[\frac{1}{\rho}\left[\underbrace{(0)\quad (0)\quad (0)\quad (0)}_{1\times 3}\quad \underset{1\times 2}{\underset{\uparrow}{(1,0)}}\right] + \frac{2}{Z_1}\,\frac{\partial Z_1}{\partial \underline{X}}\right] \tag{42}$$

$$\frac{\partial Z_6}{\partial \underline{X}} = \frac{\partial V_{EQ}}{\partial \underline{X}} = \frac{Z_6}{2Z_5}\,\frac{\partial Z_5}{\partial \underline{X}}\qquad\left(= \frac{1}{\rho_o Z_6}\,\frac{\partial Z_5}{\partial \underline{X}}\right) \tag{45}$$

$$\frac{\partial Z_7}{\partial \underline{X}} = \frac{\partial M_\infty}{\partial \underline{X}} = Z_7\ \frac{1}{Z_1}\,\frac{\partial Z_1}{\partial \underline{X}} - \frac{1}{2T}\left[\left[\underbrace{(0)\quad (0)\quad (0)\quad (0)}_{1\times 3}\quad \underset{1\times 2}{\underset{\uparrow}{(0,1)}}\right]\right] \tag{48}$$

$$\frac{\partial \rho}{\partial \underline{X}} = \left[\underbrace{(0)\quad (0)\quad (0)\quad (0)}_{1\times 3}\quad \underset{1\times 2}{\underset{\uparrow}{(1,0)}}\right] \tag{59}$$

$$\frac{\partial T}{\partial \underline{X}} = \left[\underbrace{(0)\quad (0)\quad (0)\quad (0)}_{1\times 3}\quad \underset{1\times 2}{\underset{\uparrow}{(0,1)}}\right] \tag{58}$$

$$F(T) = \left[\frac{-1.5}{T} + \frac{1 + (\log 10) \cdot 122.1 \times 10^{-(5/T)} (5/T^2)}{T + 122.1 \times 10^{-(5/T)}} \right] \tag{91}$$

$$\frac{\partial Z_8}{\partial \underline{X}} = \frac{\partial \overline{V}'}{\partial \underline{X}} = Z_8 \Bigg\{ \frac{1}{Z_7} \frac{\partial Z_7}{\partial \underline{X}} - \frac{1}{2Z_1} \frac{\partial Z_1}{\partial \underline{X}} - \frac{1}{2\rho} \frac{\partial \rho}{\partial \underline{X}}$$

$$+ \frac{1}{2} \left[\frac{1.5}{T} - \frac{1}{T + 110.4} \right] \frac{\partial T}{\partial \underline{X}} + \frac{1}{2} F(T) \frac{\partial T}{\partial \underline{X}}$$

$$- \frac{1}{2} F(T') \left[.468 \frac{\partial T}{\partial \underline{X}} + 2(3.63921 \times 10^{-5}) Z_1 \frac{\partial Z_1}{\partial \underline{X}} \right] \Bigg\} \tag{60}$$

$$\frac{\partial Z_9}{\partial \underline{X}} = \frac{\partial \alpha}{\partial \underline{X}} = \frac{1}{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}} \left(-\dot{Z}_{V|ATM,B},\ 0,\ \dot{X}_{V|ATM,B} \right) \frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}} \tag{77}$$

$$\frac{\partial Z_{10}}{\partial \underline{X}} = \frac{\partial \beta}{\partial \underline{X}} = \frac{1}{\sqrt{\dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B}} \left(\dot{X}^2_{V|ATM,B} + \dot{Y}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B} \right)}$$

$$\left(-\dot{X}_{V|ATM,B} \dot{Y}_{V|ATM,B},\ \dot{X}^2_{V|ATM,B} + \dot{Z}^2_{V|ATM,B},\ -\dot{Y}_{V|ATM,B} \dot{Z}_{V|ATM,B} \right)$$

$$\frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}} \tag{80}$$

$$\frac{\partial Z_{11}}{\partial \underline{X}} = \frac{\partial \beta'}{\partial \underline{X}} = \frac{1}{\dot{X}^2_{V|ATM,B} + \dot{Y}^2_{V|ATM,B}} \left(-\dot{Y}_{V|ATM,B},\ \dot{X}_{V|ATM,B},\ 0 \right) \frac{\partial \dot{\underline{R}}_{V|ATM,B}}{\partial \underline{X}} \tag{82}$$

$$\frac{\partial Z_{12}}{\partial \underline{X}} = \frac{\partial(\bar{q}\alpha)}{\partial \underline{X}} = \frac{\partial(Z_5 Z_9)}{\partial \underline{X}} = Z_5 \frac{\partial Z_9}{\partial \underline{X}} + Z_9 \frac{\partial Z_5}{\partial \underline{X}} \tag{84}$$

$$\frac{\partial Z_{13}}{\partial \underline{X}} = \frac{\partial(\bar{q}\beta)}{\partial \underline{X}} = \frac{\partial(Z_5 Z_{10})}{\partial \underline{X}} = Z_5 \frac{\partial Z_{10}}{\partial \underline{X}} + Z_{10} \frac{\partial Z_5}{\partial \underline{X}} \tag{86}$$

$$\frac{\partial Z_{14}}{\partial \underline{X}} = \frac{\partial(\bar{q}\beta')}{\partial \underline{X}} = \frac{\partial(Z_5 Z_{11})}{\partial \underline{X}} = Z_5 \frac{\partial Z_{11}}{\partial \underline{X}} + Z_{11} \frac{\partial Z_5}{\partial \underline{X}} \tag{88}$$

NOTE: The FMS program does not utilize the covariance matrix. Instead it inputs the error values for the state vector, ΔX_i ($i=1,\ldots 14$) and calculates RSS output errors by:

$$\Delta Z_j = \sqrt{\sum_{i=1}^{14} \left(\frac{\partial Z_j}{\partial X_i} \Delta X_i \right)^2}$$

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